

Relation connecting thermodynamics and transport of atomic unitary Fermi superfluids

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The shear viscosity is shown to be equal to the product of pressure and relaxation time in normal scale-invariant fluids, but the presence of superfluidity alters the relation. By using the mean-field BCS-Leggett theory with a gauge-invariant linear response theory for unitary Fermi superfluids, we present an explicit relation between thermodynamic quantities, including the pressure and chemical potential, and transport coefficients, including the shear viscosity, superfluid density, and anomalous shear viscosity from momentum transfer via Cooper pairs. The relation is modified when pairing fluctuations associated with noncondensed Cooper pairs are considered. Within a pairing fluctuation theory consistent with the BCS-Leggett ground state, we found an approximate relation for unitary Fermi superfluids. The exact mean-field relation and the approximate one with pairing fluctuations advance our understanding of relations between equilibrium and transport quantities in superfluids, and they help determine or constrain quantities which can be otherwise difficult to measure.

In physics, there are relations connecting different thermodynamic quantities in equilibrium, such as $P = (2/3)E$ for both noninteracting fermions [1] and unitary Fermi gases [2]. Here P is the pressure and E is the energy density, and a unitary Fermi gas consists of two-component attractive fermions on the verge of forming two-body bound states. There are also relations connecting different transport coefficients, such as the Wiedemann-Franz law showing the ratio of the thermal conductivity and the electric conductivity of normal electrons is proportional to the temperature [3]. There are, however, a third type of relations connecting thermodynamic quantities and transport coefficients. An example is the Einstein relation $D = \Gamma/(\partial n/\partial \mu)$ of Brownian motion [4], where D is the diffusion constant and is a transport coefficient, $\partial n/\partial \mu$ is the density susceptibility (which is related to the compressibility) and is a thermodynamic quantity, and Γ is a dissipative coefficient associated with the relaxation rate of the system.

It was shown [5] that the shear viscosity η is proportional to the pressure P in a normal scale-invariant fluid, implying a relation $\eta = P\tau$ with τ being the relaxation time. By using a large- N expansion and kinetic theory, the relation is shown to apply to normal unitary Fermi gases, where the divergent two-body s -wave scattering length renders the system scale invariant. The relation has been implemented in later studies of unitary Fermi gases [6]. A particular feature of this relation is that P is not a susceptibility, so it does not have the structure suggested by the fluctuation-dissipation theorem that leads to the Einstein relation [4]. There has been broad interest in transport and thermodynamic properties of ultracold atomic Fermi gases, which provide a clean testbed for analyzing strongly interacting systems and connect to

other interacting quantum systems [5–18]. Recent progresses on realizing and measuring homogeneous Fermi gases [19, 20] offer more direct check of many-body theories. The relaxation time τ naturally comes out of linear response or kinetic theories [3, 5, 16], but its measurements in cold-atoms can be challenging and past studies used the lifetime of quasi-particles as an estimation of τ [16]. The relation $\eta = P\tau$ provides a more direct means for determining τ when η and P are measured experimentally. Since the ground state of an unitary Fermi gas is a superfluid, it is imperative to investigate how the relation is modified in the superfluid phase.

Here we present a relation of superfluid unitary Fermi gases connecting the shear viscosity, pressure, superfluid density, chemical potential, and anomalous shear viscosity due to the order parameter. For the mean-field BCS-Leggett theory [21], the relation is a consequence of the equations of state and a gauge-invariant linear response theory named the consistent fluctuations of order parameter (CFOP) theory [22, 23], which may also be constructed by a path integral formalism [24]. Since strong attractive interactions can lead to preformed pairs in unitary Fermi gases [25, 26], one should distinguish condensed and noncondensed Cooper pairs. Following a pair-fluctuation theory consistent with the BCS-Leggett ground state [25, 26], the noncondensed pair contributions to the thermodynamic quantities and transport coefficients in the relation are evaluated. A modified relation resembling the one in the absence of pairing fluctuations will be presented, and numerical calculations show that the modified relation represents a reasonable approximation.

Mean-field theory: The relation connecting thermodynamic and transport quantities of unitary Fermi super-

fluids can be derived from the mean-field BCS-Leggett theory. The equations of states are given by [21] $\frac{1}{g} = \sum_{\mathbf{k}} \frac{1}{2\epsilon_{\mathbf{k}}} - \frac{m}{4\pi a} = \sum_{\mathbf{k}} \frac{1-2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}}$ and $n = \sum_{\mathbf{k}} \left[1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} + 2\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} f(E_{\mathbf{k}})\right]$, where g is the attractive coupling constant modeling the contact interaction between atoms, and a is the two-body s -wave scattering length. Here $f(x) = [\exp(x/k_B T) + 1]^{-1}$ is the Fermi distribution function, $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu = \frac{\hbar^2 k^2}{2m} - \mu$, m is the mass of the fermion, $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$ with Δ the order parameter or the energy gap. The chemical potential μ has to be self-consistently determined. Throughout the paper we will set $\hbar = 1$ and $k_B = 1$. The unitary point corresponds to $1/(k_F a) = 0$, where k_F the Fermi momentum of a noninteracting Fermi gas with the same density. Via thermodynamic arguments [2], it has been shown that $P = \frac{2}{3}E$ for unitary Fermi gases. In the Leggett-BCS theory, the following expressions satisfy the relation $P = \frac{2}{3}E$ in the superfluid phase (with a proof in the Supplementary Information):

$$P = - \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}}) - \frac{\Delta^2}{g} + 2 \sum_{\mathbf{k}} T \ln(1 + e^{-\frac{E_{\mathbf{k}}}{T}}),$$

$$E = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}}) + \frac{\Delta^2}{g} + 2 \sum_{\mathbf{k}} E_{\mathbf{k}} f(E_{\mathbf{k}}) + \mu n. \quad (1)$$

The shear viscosity is calculated based on the Kubo formalism [8, 27]:

$$\eta = - \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \text{Im} \frac{\Xi(\omega, \mathbf{q})}{\omega} = -m^2 \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \frac{\omega}{q^2} \text{Im} \chi_T(\omega, \mathbf{q})$$

$$= \frac{1}{15\pi^2 m^2} \int_0^{+\infty} dk k^6 \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2} \left(- \frac{\partial f(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}} \right) \tau. \quad (2)$$

Here $\Xi(\omega, \mathbf{q})$ is the Fourier transform of $\Xi(\bar{\tau} - \bar{\tau}', \mathbf{q}) = -i\theta(\bar{\tau} - \bar{\tau}') \langle \vec{T}^{xy}(\bar{\tau}, \mathbf{q}) \vec{T}^{xy}(\bar{\tau}', -\mathbf{q}) \rangle$, the stress tensor-stress tensor response function. $\bar{\tau}$ denotes the imaginary time, $\theta(x)$ is the Heaviside theta function, and \vec{T}^{xy} is the xy -component of the energy-momentum stress tensor \vec{T} . Formally, $\vec{T} = \frac{1}{m} (\nabla \psi_{\uparrow}^{\dagger}(\mathbf{x}) \nabla \psi_{\uparrow}(\mathbf{x}) + \nabla \psi_{\downarrow}^{\dagger}(\mathbf{x}) \nabla \psi_{\downarrow}(\mathbf{x})) + \vec{T} \mathcal{L}$, where \vec{T} is the unit tensor and \mathcal{L} is the Lagrangian density. The momentum conservation law $\frac{\partial \mathbf{J}}{\partial t} + \frac{1}{m} \nabla \cdot \vec{T} = 0$ leads to the second expression in terms of the transverse current-current correlation functions defined by $\chi_T = (\sum_{i=x}^z \chi_{JJ}^{ii} - \chi_L)/2$ with the longitudinal part given by $\chi_L = \hat{\mathbf{q}} \cdot \chi_{JJ} \cdot \hat{\mathbf{q}}$. For BCS superfluids, χ_{JJ} can be evaluated from the CFOP linear response theory [22, 23], and it has a tensor structure [22, 23] $\vec{\chi}_{JJ} = \vec{P} + \frac{n}{m} \vec{T} + \vec{C}$. Here n is the particle density, \vec{P} is the paramagnetic response function, and \vec{C} is associated with the collective modes [28, 29] but is irrelevant to the shear viscosity [30] (see Supplementary Information for more details). Here τ is the relaxation time measuring the broadening of the response functions. Moreover, η is constrained by the

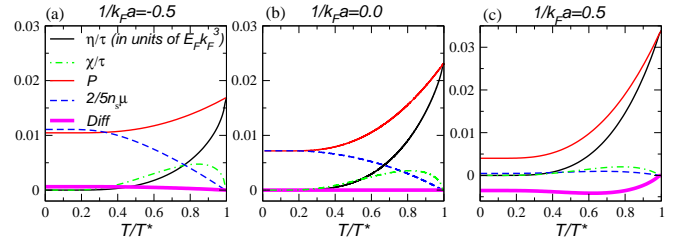


Figure 1. Plots of η/τ (black solid line), χ/τ (green dot-dash line), P (red solid line) and $\frac{2}{5}n_s\mu$ (blue dashed line) as functions of temperature (a) on the BCS side ($1/(k_F a) = -0.5$), (b) at unitary, and (c) on the BEC side ($1/(k_F a) = 0.5$). The thick pink lines denote the difference, $\text{Diff} = \eta/\tau + \chi/\tau - (P - \frac{2}{5}n_s\mu)$.

sum rule [31] $\lim_{\mathbf{q} \rightarrow 0} \int_{-\infty}^{+\infty} \frac{d\omega}{\pi} \left(- \frac{\text{Im} \chi_T(\omega, \mathbf{q})}{\omega} \right) = \frac{n_n}{m}$, where n_n is the normal fluid density. Above the mean-field critical temperature T^* , $\Delta = 0$ and Eq. (2) coincides with the expression of a normal Fermi gas [8]. In that case the shear viscosity reduces to $\eta = \frac{1}{5} n v_F \tau m^*$, where v_F is the Fermi velocity and m^* is the effective mass (details can be found in Supplementary Information).

Relation of unitary Fermi superfluids: Using the BCS-Leggett theory and the CFOP linear response theory, we found a relation connecting thermodynamic quantities and transport coefficients of a mean-field unitary Fermi superfluid:

$$\eta + \chi = (P - \frac{2}{5} \mu n_s) \tau. \quad (3)$$

The pressure P and chemical potential μ are thermodynamic quantities. On the other hand, the shear viscosity η , superfluid density n_s , and the response function χ representing the Cooper-pair contribution to momentum transfer are transport coefficients which can be inferred from linear response theory (see Supplementary Information for the derivations). The two types of quantities are connected by the relaxation time τ , which bears a similar structure as the Einstein relation.

A proof of the relation (3) is shown in Supplementary Information. The superfluid density can be obtained from the paramagnetic response function by [1, 18] $n_s = m \lim_{\omega \rightarrow 0} \lim_{\mathbf{q} \rightarrow 0} \text{Re}[P^{xx}(\omega, \mathbf{q})] + n$. The response function χ characterizes how momentum transfer can be conducted through the Cooper-pair channel in contrast to the density channel. As one will see, its derivation is almost identical to the shear viscosity except the anomalous density $\psi_{\downarrow} \psi_{\uparrow}$ replaces the normal density $\psi_{\sigma}^{\dagger} \psi_{\sigma}$ with $\sigma = \uparrow, \downarrow$. Therefore, χ is the anomalous shear viscosity from the Cooper-pair channel and it can be obtained from the $\vec{\Pi} \cdot \vec{\Pi}$ correlation function. Here $\vec{\Pi}(\mathbf{x}) = \frac{1}{m} (\nabla \psi_{\downarrow}(\mathbf{x}) \nabla \psi_{\uparrow}(\mathbf{x}) + \nabla \psi_{\uparrow}^{\dagger}(\mathbf{x}) \nabla \psi_{\downarrow}^{\dagger}(\mathbf{x}))$ is the anomalous counterpart of the stress tensor \vec{T} , and the response function is $\hat{Q}(\bar{\tau} - \bar{\tau}', \mathbf{q}) = -i\theta(\bar{\tau} - \bar{\tau}') \langle [\vec{\Pi}(\bar{\tau}, \mathbf{q}), \vec{\Pi}(\bar{\tau}', -\mathbf{q})] \rangle$. The anomalous shear viscosity

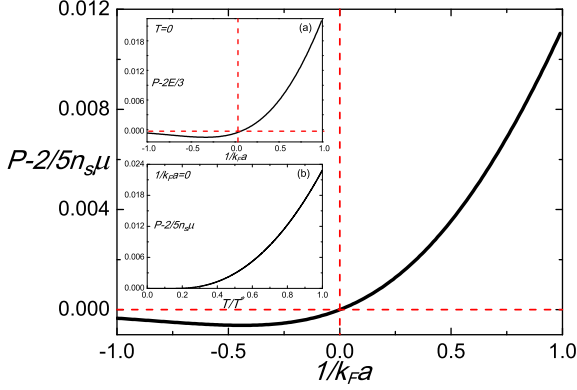


Figure 2. $P - \frac{2}{5}n_s\mu$ at $T = 0$ as a function of $1/(k_F a)$. Inset (a) shows $P - \frac{2}{3}E$ vs. $1/(k_F a)$ at zero temperature. Inset (b) shows $P - \frac{2}{5}n_s\mu$ vs. T at unitarity.

χ is defined via the $xy - xy$ component of the tensor response function \hat{Q} similar to the definition of η . Explicitly, $\chi \equiv -\lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \frac{1}{\omega} \text{Im}[Q^{xyxy}(\omega, \mathbf{q})]$. The final expressions are

$$n_s = \frac{2}{3} \sum_{\mathbf{k}} \frac{\Delta^2 k^2}{E_{\mathbf{k}}^2 m} \left[\frac{1 - 2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}} + \frac{\partial f(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}} \right],$$

$$\chi = -\frac{2\pi}{15} \sum_{\mathbf{k}} \frac{k^4}{m^2} \frac{\Delta^2}{E_{\mathbf{k}}^2} \frac{\partial f(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}} \tau. \quad (4)$$

Above T^* , n_s and χ both vanish due to the closing of the energy gap, and the relation reduces to the kinetic-theory result $\eta = P\tau$ in the normal phase. At zero temperature, η and χ both vanish since the pure BCS ground state lacks dissipation associated with transport coefficients, and the relation reduces to $P = \frac{2}{5}\mu n_s$, which is consistent with the experimental results of Ref. [20]. This zero-temperature relation remains valid when pairing fluctuations consistent with the BCS-Leggett ground state are present.

The quantities in the relation are visualized in Figure 1, where η/τ , χ/τ , P and $\frac{2}{5}n_s\mu$ as a function of temperature are shown for superfluids on the BCS side with $1/(k_F a) = -0.5$, at unitarity, and on the BEC side with $1/(k_F a) = 0.5$. Here $T^*/T_F = 0.26, 0.50, 0.84$, respectively. The relation works well at unitarity but exhibits deviations in the other cases. However, from panel (a) this relation also works well at relatively high temperature ($0.6T^* \lesssim T < T^*$) in the weak BCS regime. Although separate measurements of the normal and anomalous shear viscosity, η and χ , can be a challenge in experiments, they contribute to shear momentum transfer together and one may think of a composite shear viscosity $\tilde{\eta} \equiv \eta + \chi$ that accounts for the right hand side of the relation (3).

To better understand the deviation as the system moves away from the unitary point, we show in Figure 2 the quantity $P - \frac{2}{5}n_s\mu$ at zero temperature as a function the interaction strength $1/(k_F a)$ in the regime $-1 \leq 1/(k_F a) \leq 1$. At unitarity, the quantity vanishes as indicated by the relation (3). We also show, in the insets, $P - \frac{2}{3}E$ as a function of $1/(k_F a)$ at zero temperature and $P - \frac{2}{5}n_s\mu$ as a function of T at $1/(k_F a) = 0$. The qualitative behavior of $P - \frac{2}{5}n_s\mu$ is similar to $P - \frac{2}{3}E$ at zero temperature, which implies that the deviation is associated with the scale-invariance which guarantees certain thermodynamic relations. Moreover, $P - \frac{2}{5}n_s\mu$ is close to 0 already at low T , and this may ease its verification in experiments. Figures 1 and 2 also suggest that even though the relation (3) is not exact on the BCS side, the deviation is relatively minor and one may continue using it there as an approximation.

Beyond mean-field theory: Due to the strong attractive interactions in unitary Fermi gases, pairing fluctuation effects discerning the gap function and the order parameter should be considered [10, 23, 25, 26, 32]. Moreover, the onset temperature of pairing T^* and that of condensation (or superfluid transition temperature) T_c are different, and from t -matrix calculations T^*/T_F can be more than twice as large as T_c/T_F [26, 32]. Here we follow a t -matrix theory consistent with the BCS-Leggett ground state. The Green's function is $G(P) = [G_0^{-1}(P) - \Sigma(P)]^{-1}$, where G_0 is the noninteracting Green's function. The self energy is given by $\Sigma(P) = \sum_Q t(Q)G_0(P - Q)$, where the t -matrix $t(Q)$ can be separated into the condensed ($Q = 0$) and non-condensed ($Q \neq 0$) pair contributions $t(Q) \approx t_{sc} + t_{pg}$ with $t_{sc}(Q) = -(\Delta_{sc}^2/T)\delta(Q)$. $t_{pg}(Q) = [g^{-1} + X(Q)]^{-1}$ is constructed from $X(Q) = \sum_K G_0(Q - K)G(K)$ which consists of one bare and one full Green's functions. Here $K = (i\omega_n, \mathbf{k})$ and $Q = (i\Omega_l, \mathbf{q})$ are fermionic and bosonic four-momenta, respectively. The total gap can also be decomposed into the order parameter (from condensed pairs) Δ_{sc} and the pseudogap (from noncondensed pairs) Δ_{pg} as $\Delta^2 = \Delta_{sc}^2 + \Delta_{pg}^2$, where the pseudogap is approximated by $\Delta_{pg}^2 = -\sum_Q t_{pg}(Q)$, and the superfluid transition temperature is determined by where Δ_{sc} vanishes. The number and gap equations can be derived from $n = \sum_P G(P)$ and $\frac{1}{g} = \sum_P G(P)G_0(-P)$. The noncondensed pairs have a finite lifetime, which contributes to the relaxation time τ in linear response.

The density of grand potential, $\Omega = -P$, can be derived [10], and it leads to the equations of states after minimization. There are also contributions from a fermionic part (including condensed pairs and quasiparticles) and from a bosonic part (from noncondensed pairs). Explicitly, $P = -\Omega = -(\Omega_f + \Omega_b) = P_f + P_b$, where $\Omega_f = \Delta^2 \chi_0 + \sum_{\mathbf{k}} [(\xi_{\mathbf{k}} - E_{\mathbf{k}}) - 2T \ln(1 + e^{-\frac{E_{\mathbf{k}}}{T}})]$ and $\Omega_b = T \sum_{\mathbf{q}} \ln(1 - e^{-\frac{\Omega_{\mathbf{q}}}{T}})$. Here $\chi_0 = \frac{1}{g} - Z\mu_{\text{pair}}$, $\Omega_{\mathbf{q}} \approx \frac{q^2}{2M^*} - \mu_{\text{pair}}$, $Z = \frac{\partial X}{\partial \Omega}|_{\Omega=0, \mathbf{q}=0}$, μ_{pair} is the pair

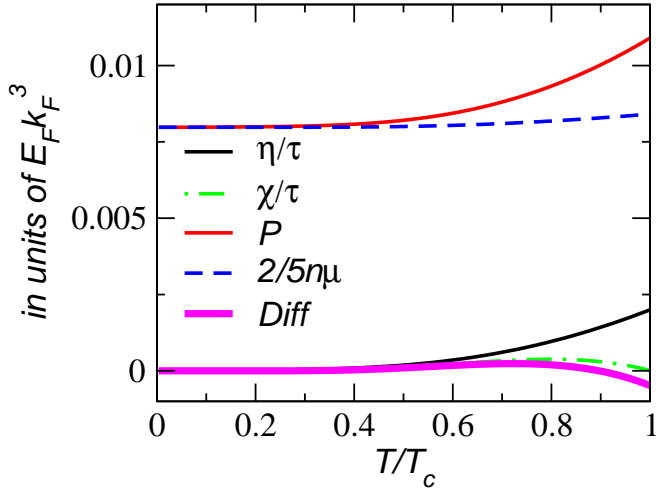


Figure 3. η/τ (black line), χ/τ (green dot-dash line), pressure P (red line), $\frac{2}{5}n\mu$ (blue dashed line) and the difference (thick pink line), and $\text{Diff}=(\eta+\chi)/\tau - (P - (2/5)n\mu)$ of a unitary Fermi superfluid as a function of temperature with pairing fluctuation effects included.

chemical potential, and $M^* = 12(\frac{\partial^2 \chi(Q)}{\partial k^2}|_{Q=0})^{-1}$ is the effective pair mass. The total energy can be decomposed in a similar fashion into $E_f = \sum_{\mathbf{k}}(\xi_{\mathbf{k}} - E_{\mathbf{k}}) + \Delta^2 \chi_0 + 2 \sum_{\mathbf{k}} E_{\mathbf{k}} f(E_{\mathbf{k}}) + \mu n$ and $E_b = \sum_{\mathbf{q}}(\Omega_{\mathbf{q}} + \mu_{\text{pair}})b(\Omega_{\mathbf{q}})$, where $b(x) = [\exp(x/T) - 1]^{-1}$ is the Bose distribution function. Similar to the mean-field theory results, for unitary Fermi gases one can show that $E_f = \frac{3}{2}P_f$ and $E_b = \frac{3}{2}P_b$, so the relation $E = \frac{3}{2}P$ still holds.

The shear viscosity can be evaluated from a modified CFOP theory with additional diagrams included to ensure the Ward identities [16, 29]. Under a weak dissipation assumption [25, 28, 29] the paramagnetic response function $\vec{P}(\omega, \mathbf{q})$ can be evaluated and the shear viscosity can be derived accordingly (see Supplementary Information). An important observation is that since the pressure P in Eq. (3) receives corrections from the noncondensed bosons, it is natural to modify the shear viscosity to acquire similar (fermionic and bosonic) corrections, $\eta = \eta_f + \eta_b$. The former comes from a calculation similar to the BCS-Leggett theory, and the latter may be approximated by the shear viscosity of a free boson gas:

$$\eta_f = \int_0^{+\infty} dk \frac{k^6}{15\pi^2 m^2} \frac{E_{\mathbf{k}}^2 - \Delta_{\text{pg}}^2}{E_{\mathbf{k}}^2} \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2} \left(-\frac{\partial f(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}} \right) \tau, \quad (5)$$

$$\eta_b = -\frac{1}{30\pi^2 M^{*2}} \int_0^{\infty} dk k^6 \frac{\partial b(\Omega_{\mathbf{k}})}{\partial \Omega_{\mathbf{k}}} \tau.$$

Since the noncondensed bosons are in local equilibrium with the fermions, it may be reasonable to assume that the relaxation time for bosons is the same as that for fermions. Moreover, we found that $\eta_b = P_b \tau$, so in this approximation the noncondensed pairs satisfy the simple relation for normal scale-invariant systems.

The anomalous shear viscosity χ is due to momentum transfer through the correlations among Cooper pairs, and the phase rigidity of superfluidity is essential. Therefore, its expression should be modified as $\chi = -\frac{2}{15} \sum_{\mathbf{k}} \frac{k^4}{m^2} \frac{\Delta_{\text{pg}}^2}{E_{\mathbf{k}}^2} \frac{\partial f(E_{\mathbf{k}})}{\partial E_{\mathbf{k}}} \tau$. This approximation also avoids possible double-counting of the shear viscosity from noncondensed pairs (already included in η_b). The right-hand side (RHS) of the relation (3) should also receive bosonic corrections. In the pairing fluctuation theory consistent with the BCS-Leggett ground state [26], there is no noncondensed pairs at zero temperature, so the zero-temperature relation $P = (2/5)\mu n_s$ should remain the same. To include the noncondensed-pair contributions at finite temperatures, we replace the superfluid density n_s on the RHS of Eq. (3) by the total number density n since the two components (superfluid and noncondensed pairs) both contribute to thermodynamics but $n_s = n$ at $T = 0$. Finally, we collect all the terms and modify the relation (3) of a unitary Fermi superfluid as

$$\tilde{\eta} \equiv \eta + \chi \approx (P - \frac{2}{5}\mu n)\tau. \quad (6)$$

To verify the validity of this approximate relation, we perform numerical calculations to check each quantity. Figure 3 shows η/τ , χ/τ , P , and $\frac{2}{5}n\mu$ as a function of temperature below T_c for a unitary Fermi superfluid, with the convention of the labels the same as Figure 1. Here $T_c/T_F \approx 0.26$, and it is known that the t -matrix approximation overestimates the transition temperature [26], which should be around $0.16T_F$ in experiments [12]. The difference $(\eta + \chi)/\tau + \frac{2}{5}n\mu - P$ is also shown. For $T/T_c \leq 0.5$, there is no observable difference, and the maximal ratio between the difference and the pressure P in the regime $0.5 \leq T/T_c \leq 1$ is less than 4.4%. Hence, the relation (6) works reasonably for unitary Fermi superfluids and may serve as a condition for constraining the relevant thermodynamic and transport quantities.

To summarize, we have presented a relation connecting thermodynamic quantities and transport coefficients of unitary Fermi superfluids. The relation involves the shear viscosity, pressure, superfluid density/particle number density, and anomalous shear viscosity characterizing momentum transfer via the Cooper pairs. The relation is exact for the mean-field BCS Leggett theory, and an approximate relation is found when pairing fluctuations from noncondensed Cooper pairs are considered, and they illustrate interesting properties of scale-invariant quantum fluids. The results of unitary Fermi superfluids also contrast different contributions and their interplays from the condensate, fermionic quasiparticles, and noncondensed pairs. The relation also helps determine the elusive relaxation time of unitary Fermi gases and constrain the relevant physical quantities.

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